

L^p - L^q boundedness of integral operators with oscillatory kernels: Linear versus quadratic phases

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Abstract

Let $T_N^{j,k} : L^p(B) \rightarrow L^q([0, 1])$ be the oscillatory integral operators defined by $T_N^{j,k} f(s) := \int_B f(x) e^{iN|x|^j s^k} dx$, $(j, k) \in \{1, 2\}^2$, where B is the unit ball in \mathbb{R}^n and $N \gg 1$. We compare the asymptotic behaviour as $N \rightarrow +\infty$ of the operator norms $\|T_N^{j,k}\|_{L^p(B) \rightarrow L^q([0, 1])}$ for all $p, q \in [1, +\infty]$. We prove that, except for the dimension $n = 1$, this asymptotic behaviour depends on the linearity or quadraticity of the phase in s only. We are led to this problem by an observation on inhomogeneous Strichartz estimates for the Schrödinger equation.

Keywords: Strichartz estimates for the Schrödinger equation, Oscillatory integrals, $L^p - L^q$ boundedness

2010 MSC: 35B45, 35Q55, 42B20

1. A remark on a counterexample to inhomogeneous Strichartz estimates for the Schrödinger equation and motivation

Consider the Cauchy problem for the inhomogeneous free Schrödinger equation with zero initial data

$$i\partial_t u + \Delta u = F(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \quad u(0, x) = 0. \quad (1)$$

Space time estimates of the form

$$\|u\|_{L_t^q(\mathbb{R}; L_x^r(\mathbb{R}^n))} \lesssim \|F\|_{L_t^{\tilde{q}'}(\mathbb{R}; L_x^{\tilde{r}'}(\mathbb{R}^n))}, \quad (2)$$

have been known as inhomogeneous Strichartz estimates. The results obtained so far (see [3, 6, 7, 10, 11]) are not conclusive when it comes to determining the optimal values of the Lebesgue exponents q, r, \tilde{q} and \tilde{r} for which the estimate (2) holds. Trying to further understand this problem, we [1]

found new necessary conditions on these exponents values. The counterexample in [1], like Example 6.10 in [3], contains an oscillatory factor with high frequency. More precisely, we used a forcing term given by

$$F(t, x) = e^{-iN^2 t} \chi_{[0, \frac{\eta}{N}]}(t) \chi_{B(\frac{\eta}{N})}(x) \quad (3)$$

where $\eta > 0$ is a fixed small number, $N \gg 1$ and $B(\frac{\eta}{N})$ is the ball with radius η/N about the origin. While in [3] the stationary phase method is applied to the inhomogeneity

$$F(t, x) = e^{-2iN^2 t^2} \chi_{[0, 1]}(t) \chi_{B(\frac{\eta}{N})}(x). \quad (4)$$

When $t \in [2, 3]$, both data in (3) and (4) force the corresponding solution $u(t, x)$ to concentrate in a spherical shell centered at the origin with radius about N . This agrees with the dispersive nature of the Schrödinger operator. The shell thickness is different in both cases though. It is about 1 in the case of the data (3) but about N in the case of (4). The necessary conditions obtained are respectively

$$\frac{1}{q} \geq \frac{n-1}{\tilde{r}} - \frac{n}{r}, \quad \frac{1}{\tilde{q}} \geq \frac{n-1}{r} - \frac{n}{\tilde{r}}$$

and

$$\left| \frac{1}{r} - \frac{1}{\tilde{r}} \right| \leq \frac{1}{n}. \quad (5)$$

Observe that the oscillatory function in (3) has a linear phase and is applied for the short time period of length $1/\sqrt{\text{frequency}}$. The oscillatory function in (4) on the other hand has a quadratic phase and the oscillation is put to work for a whole time unit. We noticed that the phase in [3] need not be quadratic and we can get the necessary condition (5) using the data

$$F_l(t, x) = e^{-iN^2 t} \chi_{[0, 1]}(t) \chi_{B(\frac{\eta}{N})}(x) \quad (6)$$

where the phase in the oscillatory function is linear. Before we show this, we recall the following approximation of oscillatory integrals according to the principle of stationary phase.

Lemma 1. (see [8], Proposition 2 Chapter VIII and Lemma 5.6 in [2])

Consider the oscillatory integral $I(\lambda) = \int_a^b e^{i\lambda\phi(s)} \psi(s) ds$. Let the phase $\phi \in C^5([a, b])$ and the amplitude $\psi \in C^3([a, b])$ such that

- (i) $\phi'(z) = 0$ for a point $z \in]a + c, b - c[$ with c a positive constant,
- (ii) $|\phi'(s)| \gtrsim 1$, for all $s \in [a, a + c] \cup [b - c, b]$,
- (iii) $|\phi''(s)| \gtrsim 1$,
- (iv) $\psi^{(j)}$ and $\phi^{(j+3)}$ are uniformly bounded on $[a, b]$ for all $j = 0, 1, 2$.

Then
$$I(\lambda) = \sqrt{\frac{2\pi}{\lambda|\phi''(z)|}} \psi(z) e^{i\lambda\phi(z) + i \operatorname{sgn}(\phi''(z)) \frac{\pi}{4}} + \mathcal{O}\left(\frac{1}{\lambda}\right),$$

where the implicit constant in the \mathcal{O} -symbol is absolute.

The norm of the inhomogeneous term F_l in (6) has the estimate

$$\|F_l\|_{L^{\vec{q}'}([0,1]; L^{\vec{r}'}(\mathbb{R}^n))} \approx \eta^{n - \frac{n}{\vec{r}}} N^{-n} N^{\frac{n}{\vec{r}}}. \quad (7)$$

For the solution of (1), we have the explicit formula

$$u(t, x) = (4\pi)^{-\frac{n}{2}} \int_0^t (t-s)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4(t-s)}} F(s, y) dy ds. \quad (8)$$

Let us estimate the solution $u_l(t, x)$ that corresponds to F_l . We shall restrict our attention to the region

$$\Omega_{\eta, N} = \{(t, x) \in [2, 3] \times \mathbb{R}^n : 2(t - 3/4)N + \eta N^{-1} < |x| < 2(t - 1/4)N - \eta N^{-1}\}.$$

It will be momentarily seen that this is the region where we can exploit Lemma 1 to approximate $u_l(t, x)$. Substituting from (6) into (8) then applying Fubini's theorem we get

$$u_l(t, x) = (4\pi)^{-\frac{n}{2}} \int_{B(\eta/N)} I_N(t, x, y) dy \quad (9)$$

where $I_N(t, x, y)$ is the oscillatory integral

$$I_N(t, x, y) = \int_0^1 e^{iN^2 \phi_N(s, t, x, y)} \psi(s, t) ds, \quad (10)$$

with the phase $\phi_N(s, t, x, y) = \frac{|x-y|^2}{4N^2} \frac{1}{t-s} - s$ and amplitude $\psi(s, t) = (t-s)^{-\frac{n}{2}}$. For simplicity, we write $\phi(\cdot)$ and $\psi(\cdot)$ in place of $\phi_N(\cdot, t, x, y)$ and $\psi(\cdot, t)$ respectively. Next, we verify the conditions (i) - (iv) for ϕ and ψ . Let $(t, x) \in \Omega_{\eta, N}$ and $y \in B(\eta/N)$. Observe then that $t - 3/4 < |x - y|/2N < t - 1/4$ and $t - s \in [1, 3]$. Therefore

(i) If z is such that $\phi'(z) = 0$ then $z = t - |x - y|/2N$. Moreover, $z \in]1/4, 3/4[$.

(ii) ϕ' is monotonically increasing so $\min_{s \in [0,1]} \phi'(s) = \phi'(0) = \frac{|x - y|^2}{4N^2 t^2} >$

$$\left(1 - \frac{3}{4t}\right)^2 \gtrsim 1.$$

(iii) $\phi''(s) = \frac{|x - y|^2}{2N^2} \frac{1}{(t - s)^3} \approx 1.$

(iv) $\phi^{(j)}(s) = \frac{|x - y|^2}{4N^2} \frac{j!}{(t - s)^{(j+1)}} \approx 1, j = 3, 4, 5, \quad \psi(s) = (t - s)^{-\frac{n}{2}} \approx 1,$

$$\psi'(s) = \frac{n}{2}(t - s)^{-\frac{n}{2}-1} \approx 1, \quad \psi''(s) = \frac{n}{2}(\frac{n}{2} + 1)(t - s)^{-\frac{n}{2}-2} \approx 1.$$

Now, applying Lemma 1 to the oscillatory integral $I_N(t, x, y)$ in (10) yields

$$I_N(t, x, y) = \sqrt{\frac{2\pi}{\phi_N''(z, t, x, y)}} \psi(z, t) \frac{e^{\frac{\pi}{4}i}}{N} e^{iN^2 \phi_N(z, t, x, y)} + \mathcal{O}\left(\frac{1}{N^2}\right). \quad (11)$$

Since $\phi_N(z, t, x, y) + t = |x - y|/N$ and since $N(|x - y| - |x|) = \mathcal{O}(\eta)$ whenever

$(t, x) \in \Omega_{\eta, N}, y \in B(\eta/N)$. Then $N^2 \phi_N(z, t, x, y) + N^2 t = N|x| + \mathcal{O}(\eta)$. Hence

$$e^{iN^2 \phi_N(z, t, x, y)} = e^{i(N|x| - N^2 t)} e^{\mathcal{O}(\eta)} = e^{i(N|x| - N^2 t)} (1 + \mathcal{O}(\eta)). \quad (12)$$

Inserting (12) into (11) then returning to (9), we discover

$$u_l(t, x) = \frac{(4\pi)^{\frac{1-n}{2}}}{\sqrt{2}} \frac{e^{\frac{\pi}{4}i}}{N} e^{i(N|x| - N^2 t)} \int_{B(\eta/N)} \frac{\psi(z, t)}{\sqrt{\phi_N''(z, t, x, y)}} (1 + \mathcal{O}(\eta)) dy \\ + \mathcal{O}\left(\frac{1}{N^2}\right) \int_{B(\eta/N)} dy.$$

Recalling that $\psi, \phi'' \approx 1$, we immediately deduce the estimate

$$|u_l(t, x)| \gtrsim \frac{|B(\eta/N)|}{N} \approx \eta^n N^{-(1+n)}, \quad (t, x) \in \Omega_{\eta, N}. \quad \text{Thus, for all } t \in [2, 3],$$

$$\|u_l(t, x)\|_{L_x^r(\mathbb{R}^n)} \geq \left(\int_{2(t-3/4)N + \eta N^{-1} < |x| < 2(t-1/4)N - \eta N^{-1}} |u_l(t, x)|^r dx \right)^{\frac{1}{r}} \gtrsim \eta^n N^{-(1+n) + \frac{n}{r}}.$$

Consequently

$$\|u_l\|_{L_t^q(\mathbb{R}; L_x^r(\mathbb{R}^n))} \geq \|u_l\|_{L_t^q([2,3]; L_x^r(\mathbb{R}^n))} \gtrsim \eta^n N^{-(1+n)+\frac{n}{r}}. \quad (13)$$

Lastly, it follows from (7) and (13) that

$$\|u_l\|_{L_t^q(\mathbb{R}; L_x^r(\mathbb{R}^n))} / \|F_l\|_{L^{\tilde{q}'}([0,1]; L^{\tilde{r}'}(\mathbb{R}^n))} \gtrsim \eta^{\frac{n}{\tilde{r}}} N^{\frac{n}{\tilde{r}} - \frac{n}{r} - 1}$$

which, for a fixed η , blows up as $N \rightarrow +\infty$ if $\frac{n}{r} - \frac{n}{\tilde{r}} > 1$. In the light of duality this implies the necessary condition (5).

These examples made us wonder how exactly different are linear oscillations from quadratic ones if we capture the cancellations in Lebesgue spaces. One way to see this is to consider the operators $T_N^{j,k} : L^p(B) \rightarrow L^q([0,1])$ defined by

$$T_N^{j,k} f(s) := \int_B f(x) e^{iN|x|^j s^k} dx, \quad (j, k) \in \{1, 2\}^2, \quad (14)$$

where B is the unit ball in \mathbb{R}^n , and compare the asymptotic behaviour as $N \rightarrow +\infty$ of their operator norms for all $p, q \in [1, +\infty]$. Let $C_{j,k,n} : [0, 1]^2 \rightarrow \mathbb{R}$ be the functions defined by

$$C_{j,k,n} \left(\frac{1}{p}, \frac{1}{q} \right) := \alpha \quad \text{if} \quad \|T_N^{j,k}\|_{L^p(B) \rightarrow L^q([0,1])} \approx N^{-\alpha}.$$

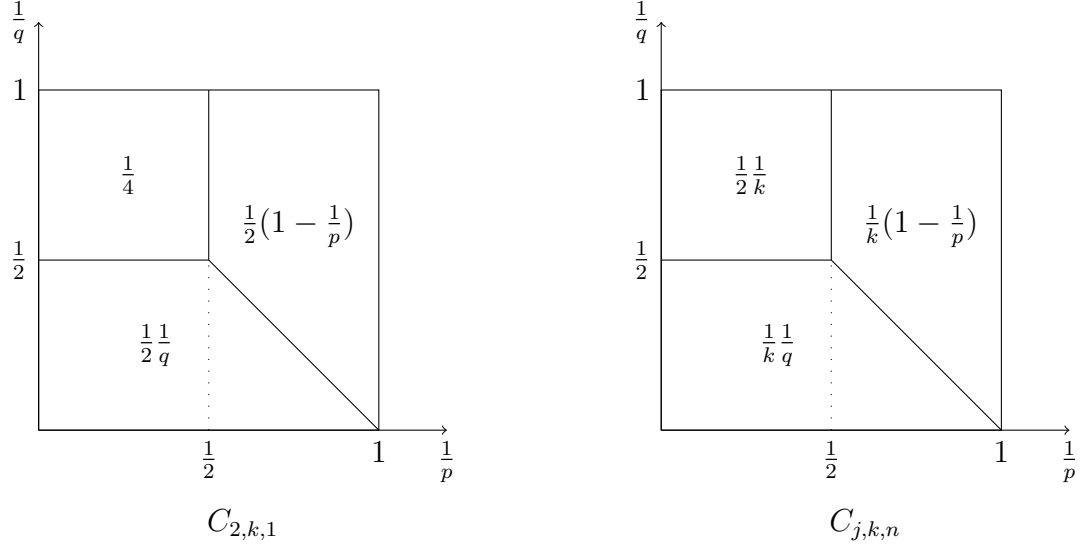
We discover that $C_{j,k,n}$ is a continuous function with range $[0, 1/4]$ when $n = 1, j = 2$ and $[0, 1/2k]$ otherwise (see the figure below). We actually prove that

Theorem 2.

$$C_{j,k,n} \left(\frac{1}{p}, \frac{1}{q} \right) = \begin{cases} \frac{1}{4} \sigma \left(\frac{1}{p}, \frac{1}{q} \right), & n = 1, j = 2; \\ \frac{1}{2k} \sigma \left(\frac{1}{p}, \frac{1}{q} \right), & n \geq j. \end{cases}$$

where

$$\sigma(a, b) := \begin{cases} 2b, & 0 \leq a \leq 1 - b, \quad 0 \leq b \leq \frac{1}{2}; \\ 2(1 - a), & \frac{1}{2} \leq a \leq 1, \quad a + b \geq 1; \\ 1, & 0 \leq a \leq \frac{1}{2}, \quad \frac{1}{2} \leq b \leq 1. \end{cases} \quad (15)$$



Remark 1. For each $p, q \in [1, \infty]$, and all dimension $n > 1$, the asymptotic behaviour of $\|T_N^{j,k}\|_{L^p(B) \rightarrow L^q([0,1])}$ as $n \rightarrow +\infty$ is determined only by the linearity or quadraticity of the phase in s . The role of the power j of x appears exclusively in the dimension $n = 1$.

Remark 2. There is nothing special about neither the unit interval nor the unit ball in defining the operators $T_N^{j,k}$. Actually we shall make use of Hölder inclusions of L^p spaces on measurable sets of finite measure (see Lemma 3 below). So we may take any suitable two such sets provided their finite measures are asymptotically equivalent to a constant independent of N as $N \rightarrow +\infty$.

Foschi [2] studied a discrete version of an operator a little simpler than the integral operator $T_N^{1,1}$. He considered the operator $D_N : \ell^p(\mathbb{C}^N) \rightarrow L^q(-\pi, \pi)$ that assigns to each vector $a = (a_0, a_1, \dots, a_{N-1}) \in \mathbb{C}^N$ the 2π -periodic trigonometric polynomial $D_N a(t) = \sum_{m=0}^{N-1} a_m e^{im t}$ and described the asymptotic behaviour of $\sup_{a \in \mathbb{C}^N - \{0\}} \|D_N a\|_{L^q([-\pi, \pi])} / \|a\|_{\ell^p(\mathbb{C}^N)}$ as $N \rightarrow$

$+\infty$, for all $1 \leq p, q \leq +\infty$. The norms there are defined by

$$\begin{aligned} \|a\|_{\ell^p} &= \left(\sum_{m=0}^{N-1} |a_m|^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, & \|a\|_{\ell^\infty} &= \max_{0 \leq m \leq N-1} |a_m|, \\ \|f\|_{L^q} &= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^q dt \right)^{\frac{1}{q}}, \quad 1 \leq q < \infty, & \|f\|_{L^\infty} &= \max_{|t| \leq \pi} |f(t)|. \end{aligned}$$

This was followed by a similar investigation (see Section 5 in [2]) of a linear integral operator with an oscillatory kernel $L_N : L^p([0, 1]) \rightarrow L^q([0, 1])$ defined by

$$L_N f(t) := \int_0^1 e^{iN/(1+t+s)} \frac{f(s)}{(1+t+s)^\gamma} ds, \quad \text{for some fixed } \gamma \geq 0.$$

2. Proof of Theorem 2

In order to show Theorem 2, we shall go through the following steps.

Step 1. Find lower bounds for $\|T_N^{j,k}\|_{L^p(B) \rightarrow L^q([0,1])}$ for all $p, q \in [1, +\infty]$: Test the ratio $\|T_N^{j,k} f\|_{L^q([0,1])} / \|f\|_{L^p(B)}$ for functions $f \in L^p(B)$ that kill or at least slow down the oscillations in the integrals $T_N^{j,k} f$. Of course this ratio is majorized by $\|T_N^{j,k}\|_{L^p(B) \rightarrow L^q([0,1])} = \sup_{f \in L^p(B) - \{0\}} \|T_N^{j,k} f\|_{L^q([0,1])} / \|f\|_{L^p(B)}$.

But what is really interesting is the fact that such functions likely maximize the ratio as well.

Step 2. We find upper bounds for $\|T_N^{j,k}\|_{L^p(B) \rightarrow L^q([0,1])}$ for all $p, q \in [1, +\infty]$. Thanks to interpolation and Hölder's inequality, we merely need an upper bound for $\|T_N^{j,k}\|_{L^2(B) \rightarrow L^2([0,1])}$.

Lemma 3. Let $T_N^{j,k} : L^p(B) \rightarrow L^q([0, 1])$ be as in (14). Assume that

$$\|T_N^{j,k} f\|_{L^2([0,1])} \leq c_{j,k,N} \|f\|_{L^2(B)}. \quad (16)$$

Then

$$\|T_N^{j,k}\|_{L^p(B) \rightarrow L^q([0,1])} \lesssim_{p,q,n} c_{j,k,N}^{\sigma(\frac{1}{p}, \frac{1}{q})} \quad (17)$$

where $\sigma : [0, 1]^2 \rightarrow [0, 1]$ is the continuous function in (15).

Proof. If we take absolute values of both sides of (14) we get the trivial estimate

$\| T_N^{j,k} f \|_{L^\infty([0,1])} \leq \| f \|_{L^1(B)}$. Interpolating this with (16) using Riesz-Thorin theorem ([4]) implies

$$\| T_N^{j,k} f \|_{L^q([0,1])} \leq c_{j,k,N}^{2(1-\frac{1}{p})} \| f \|_{L^p(B)}, \quad \frac{1}{2} \leq \frac{1}{p} \leq 1, \quad \frac{1}{q} = 1 - \frac{1}{p}. \quad (18)$$

Since, by Hölder's inequality, $\| T_N^{j,k} f \|_{L^{\bar{q}}([0,1])} \leq \| T_N^{j,k} f \|_{L^q([0,1])}$ whenever $1 \leq \bar{q} \leq q \leq \infty$, then

$$\| T_N^{j,k} f \|_{L^q([0,1])} \leq c_{j,k,N}^{2(1-\frac{1}{p})} \| f \|_{L^p(B)}, \quad \frac{1}{2} \leq \frac{1}{p} \leq 1, \quad 1 - \frac{1}{p} \leq \frac{1}{q} \leq 1. \quad (19)$$

Applying Hölder's inequality once more we find that if $1 \leq p \leq \bar{p} \leq \infty$, then

$$\| f \|_{L^p(B)} \leq |B|^{\frac{1}{p}-\frac{1}{\bar{p}}} \| f \|_{L^{\bar{p}}(B)}. \text{ Therefore by (18) we have}$$

$$\| T_N^{j,k} f \|_{L^q([0,1])} \leq |B|^{1-\frac{1}{p}-\frac{1}{q}} c_{j,k,N}^{2/q} \| f \|_{L^p(B)}, \quad 0 \leq \frac{1}{q} \leq \frac{1}{2}, \quad 0 \leq \frac{1}{p} \leq 1 - \frac{1}{q}. \quad (20)$$

Moreover, since we know from (19) that

$$\| T_N^{j,k} f \|_{L^q([0,1])} \leq c_{j,k,N} \| f \|_{L^2(B)}, \quad \frac{1}{2} \leq \frac{1}{q} \leq 1, \quad \text{then}$$

$$\| T_N^{j,k} f \|_{L^q([0,1])} \leq |B|^{\frac{1}{2}-\frac{1}{p}} c_{j,k,N} \| f \|_{L^p(B)}, \quad 0 \leq \frac{1}{p} \leq \frac{1}{2}, \quad \frac{1}{2} \leq \frac{1}{q} \leq 1. \quad (21)$$

If the constants in inequalities (18) - (21) were sharp, they would be precisely the values of the corresponding norms $\| T_N^{j,k} \|_{L^p(B) \rightarrow L^q([0,1])}$. Unfortunately, we are not able to compute the optimal constant $c_{j,k,N}$ in the energy estimate (16). Nevertheless, the constants $c_{j,k,N}^{\sigma(\frac{1}{p}, \frac{1}{q})}$ in (17) would be good enough for our purpose if, for each $p, q \in [1, +\infty]$, they were asymptotically equivalent, as $N \rightarrow +\infty$, to the corresponding lower bounds of $\| T_N^{j,k} \|_{L^p(B) \rightarrow L^q([0,1])}$ that we compute in *Step 1*.

Step 1.

(i) **Focusing data**

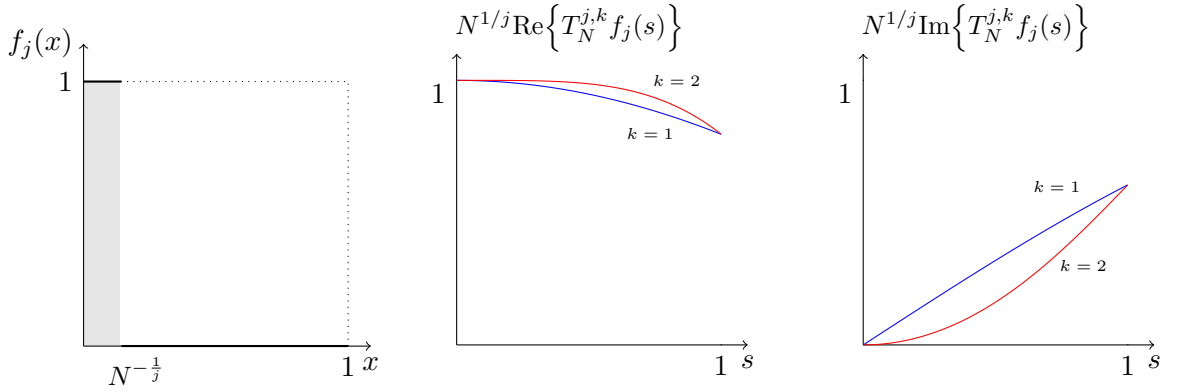
When $x \in B(\eta/N^{\frac{1}{j}})$ we have $e^{iN|x|^j s^k} = e^{\mathcal{O}(\eta)} = 1 + \mathcal{O}(\eta)$, for all $s \in [0, 1]$. Thus, if we take f_j to be the focusing functions $f_j = \chi_{B(\eta/N^{\frac{1}{j}})}$ then $\|f\|_{L^p(B)} = |B(\eta/N^{\frac{1}{j}})|^{\frac{1}{p}}$ and

$$T_N^{j,k} f_j(s) = \int_{B(\eta/N^{\frac{1}{j}})} e^{iN|x|^j s^k} dx = \int_{B(\eta/N^{\frac{1}{j}})} (1 + \mathcal{O}(\eta)) dx \gtrsim |B(\eta/N^{\frac{1}{j}})|$$

for all $0 \leq s \leq 1$. Consequently, since η is fixed,

$$\|T_N^{j,k}\|_{L^p(B) \rightarrow L^q([0,1])} \geq \frac{\|T_N^{j,k} f_j\|_{L^q([0,1])}}{\|f_j\|_{L^p(B)}} \gtrsim N^{-\frac{n}{j}(1-\frac{1}{p})}. \quad (22)$$

The figure below illustrates the one dimensional case.



Both real and imaginary parts of the functions $T_N^{1,k} f_1$ and $T_N^{2,k} f_2$ have the same profile.

(ii) **Constant data**

Let $g(x) = 1$. Whenever $s \in [0, \eta/N^{\frac{1}{k}}]$ we have $iN|x|^j s^k = \mathcal{O}(\eta)$ for all $x \in B$ and it follows that $e^{iN|x|^j s^k} = 1 + \mathcal{O}(\eta)$. Hence, when $s \in [0, \eta/N^{\frac{1}{k}}]$,

$$T_N^{j,k} g(s) = \int_B e^{iN|x|^j s^k} dx = \int_B (1 + \mathcal{O}(\eta)) dx \gtrsim 1.$$

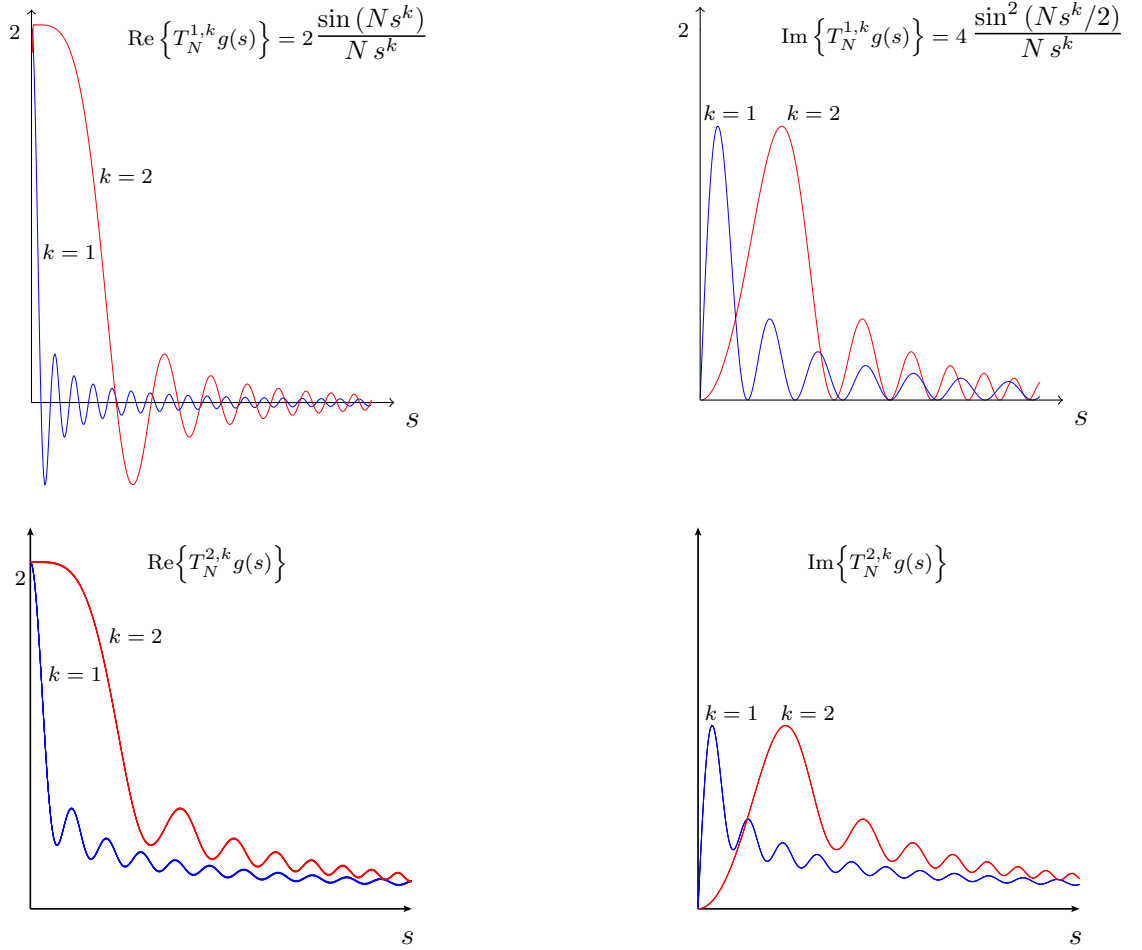
Therefore, recalling that η is fixed,

$$\int_0^1 |T_N^{j,k} g(s)|^q ds \geq \int_0^{\eta/N^{\frac{1}{k}}} |T_N^{j,k} g(s)|^q ds \gtrsim \int_0^{\eta/N^{\frac{1}{k}}} ds \approx N^{-\frac{1}{k}}. \quad (23)$$

In view of (23), we deduce that

$$\| T_N^{j,k} \|_{L^p(B) \rightarrow L^q([0,1])} \geq \frac{\| T_N^{j,k} g \|_{L^q([0,1])}}{\| g \|_{L^p(B)}} \gtrsim N^{-\frac{1}{k} \frac{1}{q}}. \quad (24)$$

By rescaling, it is easy to verify that the estimate (24) follows for any complex-valued constant function g . The figure below shows the behaviour of $T_N^{j,k} g$ on $[0, 1]$ in the dimension $n = 1$.



Functions $\text{Re}\{T_N^{1,k} g(s)\}$ vanish and $\text{Re}\{T_N^{2,k} g(s)\}$ change monotonicity, for the first time, when $s = \sqrt[k]{\pi/N}$

(iii) Oscillatory data

Consider the oscillatory function $h(x) = e^{2iN(|x|^2 - |x|)}$. Using polar coordi-

nates we can write

$$T_N^{j,k} h(s) = \int_{S^{n-1}} \int_0^1 e^{iN(\rho^j s^k + 2\rho^2 - 2\rho)} \rho^{n-1} d\rho d\omega = \omega_{n-1} I_N^{j,k}(s)$$

where $I_N^{j,k}(s)$ is the oscillatory integral given by

$$I_N^{j,k}(s) = \int_0^1 e^{iN\phi_{j,k}(\rho;s)} \rho^{n-1} d\rho \quad (25)$$

with the phase $\phi_{j,k}(\rho;s) = \rho^j s^k + 2\rho^2 - 2\rho$.

The quadratic function $\rho \rightarrow \phi_{j,k}(\rho;s)$, after a suitable translation along the vertical axis, has a single nondegenerate stationary point that happens to lie well inside $] \frac{1}{5}, \frac{4}{5} [$. Indeed, one can simply write

$$\phi_{j,k}(\rho;s) = \begin{cases} 2\left(\rho - \frac{2-s^k}{4}\right)^2 - \frac{(2-s^k)^2}{8}, & j=1; \\ (2+s^k)\left(\rho - \frac{1}{2+s^k}\right)^2 - \frac{1}{(2+s^k)^2}, & j=2. \end{cases}$$

Notice also that $(2-s^k)/4 \in [\frac{1}{4}, \frac{1}{2}]$ and $(2+s^k)^{-1} \in [\frac{1}{3}, \frac{1}{2}]$ when $s \in [0, 1]$. In fact, this is what we were after when we used the oscillatory function h with its particular quadratic phase. Let us see how we benefit from this. We shall work on the integral $I_N^{1,k}(s)$ and the applicability of the same procedure to the integral $I_N^{2,k}(s)$ will be obvious. For simplicity, let z denote $(2-s^k)/4$. Then

$$\begin{aligned} e^{2iNz^2} I_N^{1,k}(s) &= \int_0^1 e^{2iN(\rho-z)^2} \rho^{n-1} d\rho \\ &= z^{n-1} \int_0^1 e^{2iN(\rho-z)^2} d\rho + \int_0^1 e^{2iN(\rho-z)^2} (\rho^{n-1} - z^{n-1}) d\rho. \end{aligned} \quad (26)$$

We compute

$$\int_0^1 e^{2iN(\rho-z)^2} d\rho = \int_{-\infty}^{+\infty} e^{2iN(\rho-z)^2} d\rho - \int_{-\infty}^0 e^{2iN(\rho-z)^2} d\rho - \int_1^{+\infty} e^{2iN(\rho-z)^2} d\rho. \quad (27)$$

Using the identity (See Exercise 2.26 in [9])

$$\int_{-\infty}^{+\infty} e^{-ax^2} e^{bx} dx = \sqrt{\frac{\pi}{a}} e^{b^2/4a}, \quad a, b \in \mathbb{C}, \operatorname{Re}(a) > 0 \quad \text{we get}$$

$$\int_{-\infty}^{+\infty} e^{2iN(\rho-z)^2} d\rho = \sqrt{\frac{\pi}{2N}} e^{\frac{\pi}{4}i}. \quad (28)$$

And since

$$\left| \int_{-\infty}^0 e^{2iN(\rho-z)^2} \partial_\rho (\rho-z)^{-1} d\rho \right| \leq \frac{1}{z}, \quad \left| \int_1^{+\infty} e^{2iN(\rho-z)^2} \partial_\rho (\rho-z)^{-1} d\rho \right| \leq \frac{1}{1-z},$$

then integration by parts implies

$$\int_{-\infty}^0 e^{2iN(\rho-z)^2} d\rho = \frac{i e^{2iNz^2}}{4Nz} + \mathcal{O}\left(\frac{1}{Nz}\right), \quad (29)$$

$$\int_1^{+\infty} e^{2iN(\rho-z)^2} d\rho = \frac{i e^{2iN(1-z)^2}}{4N(1-z)} + \mathcal{O}\left(\frac{1}{N(1-z)}\right). \quad (30)$$

Recalling that $\frac{1}{4} \leq z \leq \frac{1}{2}$ and using (28), (29), (30) in (27) we obtain

$$\int_0^1 e^{2iN(\rho-z)^2} d\rho = \sqrt{\frac{\pi}{2N}} e^{\frac{\pi}{4}i} + \mathcal{O}\left(\frac{1}{N}\right). \quad (31)$$

This gives us an estimate for the first integral on the right hand side of (26). The second integral is $\mathcal{O}(1/N)$. This follows from integration by parts and the smoothness of the polynomial $P(\rho; z) := (\rho^{n-1} - z^{n-1})/(\rho - z) = \sum_{\ell=0}^{n-2} \rho^{n-2-\ell} z^\ell$ as we can write

$$\int_0^1 e^{2iN(\rho-z)^2} (\rho^{n-1} - z^{n-1}) d\rho = \frac{1}{4iN} \int_0^1 P(\rho; z) \partial_\rho e^{2iN(\rho-z)^2} d\rho.$$

Plugging (31) together with the latter estimate into (26) we get that

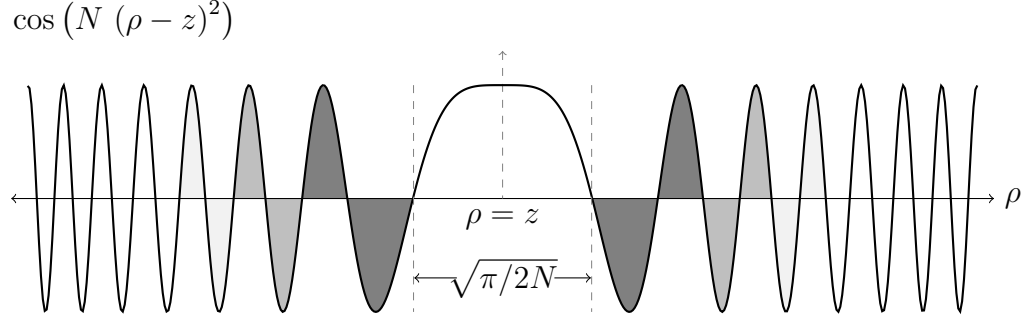
$$e^{2iNz^2} I_N^{1,k}(s) = z^{n-1} \sqrt{\frac{\pi}{2N}} e^{\frac{\pi}{4}i} + \mathcal{O}\left(\frac{1}{N}\right). \quad (32)$$

From (32) follows the estimate

$$\left| I_N^{1,k}(s) \right| \gtrsim N^{-1/2}.$$

An explanation for the estimate above comes from the fact that the function $\lambda_N(\rho; z) = \cos(2N(\rho-z)^2)$ remains positive for $|\rho-z| < \sqrt{(\pi/4N)}$ and the further we move from the stationary point $\rho = z$ it, unlike the slowly

varying factor ρ^{n-1} , oscillates rapidly for large N so that, when summing over ρ , integrals over neighbouring halfwaves where λ_N changes sign almost cancel. See the figure below. An identical estimate for $I_N^{2,k}(s)$ follows applying the same argument above. The approach adopted here is standard. It represents the key idea of the proof of the stationary phase method illustrated by Lemma 1.



Finally, since $\|h\|_{L^p(B)} = |B|^{1/p} \approx 1$, then

$$\|T_N^{j,k}\|_{L^p(B) \rightarrow L^q([0,1])} \geq \frac{\|T_N^{j,k}h\|_{L^q([0,1])}}{\|h\|_{L^p(B)}} \gtrsim N^{-\frac{1}{2}}. \quad (33)$$

Putting (22), (24) and (33) together we deduce

$$\|T_N^{j,k}\|_{L^p(B) \rightarrow L^q([0,1])} \gtrsim N^{-\min\{\frac{n}{j}(1-\frac{1}{p}), \frac{1}{k}\frac{1}{q}, \frac{1}{2}\}} = N^{-C_{j,k,n}(\frac{1}{p}, \frac{1}{q})}.$$

Step 2. The $L^2 - L^2$ estimate takes the form:

$$\left. \begin{aligned} \|T_N^{j,k}f\|_{L^2([0,1])} &\lesssim N^{-1/2k} \|f\|_{L^2(B)}, \quad n \geq j, \\ \|T_N^{2,k}f\|_{L^2([0,1])} &\lesssim N^{-n/2j} \|f\|_{L^2(B)}, \quad n = 1. \end{aligned} \right\} \quad (34)$$

Besides (24), the estimate (34) demonstrates the difference between linear ($k = 1$) and quadratic ($k = 2$) oscillations. Let $x \in \mathbb{R}^n - \{0\}$. The phase $s \rightarrow |x|^j s^k$ of the oscillatory factor in (14) is non-stationary when $k = 1$. While in the case $k = 2$, it is stationary with the nondegenerate critical point $s = 0$. This is where non-stationary and stationary phase methods (see lemmas 4 and 5 below) for estimating oscillatory integrals come into play. As expected from (22), the role of j appears only in the dimension $n = 1$. Using the estimate (34) in Lemma 3 we infer

$$\|T_N^{j,k}\|_{L^p(B) \rightarrow L^q([0,1])} \lesssim N^{-C_{j,k,n}(\frac{1}{p}, \frac{1}{q})}.$$

3. Proof of the energy estimate (34)

To prove the estimate (34) we need lemmas 6, 7 and 9 that we give below. Lemma 6 is based on the assertions of lemmas 4 and 5.

Lemma 4. ([8], Proposition 1 Chapter VIII) Let $\psi \in C_c^\infty(\mathbb{R})$ and let $I(\lambda) = \int_{\mathbb{R}} \psi(s) e^{i\lambda s} ds$. Then $|I(\lambda)| \lesssim \min \left\{ \frac{1}{1+|\lambda|}, \frac{1}{1+\lambda^2} \right\}$.

Observing that $\int_0^1 e^{i\lambda s^2} ds = \frac{1}{2} \int_{-1}^1 e^{i\lambda s^2} ds$ and arguing as in (27)-(31) implies the estimate in Lemma 5.

Lemma 5.

$$\left| \int_0^1 e^{i\lambda s^2} ds \right| \lesssim \max \left\{ \frac{1}{1+\sqrt{|\lambda|}}, \frac{1}{1+|\lambda|} \right\}.$$

Lemma 6. Let $\psi \in C_c^\infty(\mathbb{R})$ and let $K_N^{j,k} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}$ be defined by

$$K_N^{j,k}(x, y) := \begin{cases} \int_{\mathbb{R}} \psi(s) e^{iN(|x|^j - |y|^j)s} ds, & k = 1; \\ \int_0^1 e^{iN(|x|^j - |y|^j)s^2} ds, & k = 2. \end{cases}$$

Then

$$|K_N^{j,1}(x, y)| \lesssim \min \left\{ \left(1 + N \left| |x|^j - |y|^j \right| \right)^{-1}, \left(1 + N^2 \left(|x|^j - |y|^j \right)^2 \right)^{-1} \right\}, \quad (35)$$

$$|K_N^{j,2}(x, y)| \lesssim \max \left\{ \left(1 + \sqrt{N} \sqrt{\left| |x|^j - |y|^j \right|} \right)^{-1}, \left(1 + N \left| |x|^j - |y|^j \right| \right)^{-1} \right\}. \quad (36)$$

The next lemma is mainly a consequence of Young's inequality.

Lemma 7. Let $p, q, r \geq 1$ and $1/p + 1/q + 1/r = 2$. Let $f \in L^p(B)$, $g \in L^q(B)$ and $h \in L^r([0, 1])$. Then

$$\left| \int_B \int_B f(x) f(y) h(|x|^m - |y|^m) dx dy \right| \lesssim \|f\|_{L^p(B)} \|g\|_{L^q(B)} \|h\|_{L^r([0,1])}$$

provided $m \leq n$.

Proof. Switching to polar coordinates by setting $x = r_1\theta_1$ and $y = r_2\theta_2$ then applying Fubini's theorem gives

$$\left| \int_B \int_B f(x) f(y) h(|x|^m - |y|^m) dx dy \right| \leq \int_{S^{n-1}} \int_{S^{n-1}} |Q(\theta_1, \theta_2)| d\theta_1 d\theta_2 \quad (37)$$

where

$$Q(\theta_1, \theta_2) = \int_0^1 \int_0^1 f(r_1\theta_1) g(r_2\theta_2) h(r_1^m - r_2^m) r_1^{n-1} r_2^{n-1} dr_1 dr_2.$$

Changing variables $r_i^m \rightarrow \rho_i$ then using Young's inequality we get

$$|Q(\theta_1, \theta_2)| \lesssim \left(\int_0^1 |f(\sqrt[m]{\rho_1} \theta_1)|^p \rho_1^{p \frac{n-m}{m}} d\rho_1 \right)^{\frac{1}{p}} \left(\int_0^1 |g(\sqrt[m]{\rho_2} \theta_2)|^q \rho_2^{q \frac{n-m}{m}} d\rho_2 \right)^{\frac{1}{q}} \|h\|_{L^r([0,1])}.$$

Reversing the variables change in the first two integrals on the right-hand side of the latter estimate we obtain

$$\begin{aligned} |Q(\theta_1, \theta_2)| &\lesssim \left(\int_0^1 |f(r_1 \theta_1)|^p r_1^{(p-1)(n-m)} r_1^{n-1} dr_1 \right)^{\frac{1}{p}} \\ &\quad \left(\int_0^1 |g(r_2 \theta_2)|^q r_2^{(p-1)(n-m)} r_2^{n-1} dr_2 \right)^{\frac{1}{q}} \|h\|_{L^r([0,1])} \\ &\leq \left(\int_0^1 |f(r_1 \theta_1)|^p r_1^{n-1} dr_1 \right)^{\frac{1}{p}} \left(\int_0^1 |g(r_2 \theta_2)|^q r_2^{n-1} dr_2 \right)^{\frac{1}{q}} \|h\|_{L^r([0,1])} \end{aligned} \quad (38)$$

as long as $m \leq n$. Invoking Hölder's inequality it follows that

$$\begin{aligned} &\int_{S^{n-1}} \left(\int_0^1 |f(r_1 \theta_1)|^p r_1^{n-1} dr_1 \right)^{\frac{1}{p}} d\theta_1 \\ &\leq \omega_{n-1}^{1-\frac{1}{p}} \left(\int_{S^{n-1}} \int_0^1 |f(r_1 \theta_1)|^p r_1^{n-1} dr_1 d\theta_1 \right)^{\frac{1}{p}} = \omega_{n-1}^{1-\frac{1}{p}} \|f\|_{L^p(B)}, \end{aligned} \quad (39)$$

$$\begin{aligned} &\int_{S^{n-1}} \left(\int_0^1 |g(r_2 \theta_2)|^q r_2^{n-1} dr_2 \right)^{\frac{1}{q}} d\theta_2 \\ &\leq \omega_{n-1}^{1-\frac{1}{q}} \left(\int_{S^{n-1}} \int_0^1 |g(r_2 \theta_2)|^q r_2^{n-1} dr_2 d\theta_2 \right)^{\frac{1}{q}} = \omega_{n-1}^{1-\frac{1}{q}} \|g\|_{L^q(B)}. \end{aligned} \quad (40)$$

Returning to (37) with the estimates (38), (39) and (40) concludes the proof. \square

Remark 3 together with Lemma 8 are needed to show Lemma 9.

Remark 3. *Suppose that the integral*

$$J = \int_{-b_1}^{b_1} \dots \int_{-b_m}^{b_m} K(t_1, \dots, t_m) f_1(t_1) \dots f_m(t_m) dt_1 \dots dt_m$$

exists. If K is even in all its variables then

$$J = \int_0^{b_1} \dots \int_0^{b_m} K(t_1, \dots, t_m) \prod_{i=1}^m (f_i(t_i) + f_i(-t_i)) dt_1 \dots dt_m.$$

This follows easily from the fact that the integrand in the second expression for J is even in all variables.

Lemma 8 discusses the boundedness of a bilinear form with a homogeneous kernel.

Lemma 8. *Let $f \in L^p([0, 1])$ and $g \in L^q([0, 1])$ with $1 \leq p \leq +\infty$ and $1/p + 1/q = 1$. Assume that $K : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ is homogeneous of degree -1 , that is, $K(\lambda x, \lambda y) = \lambda^{-1} K(x, y)$, for $\lambda > 0$. Assume also that*

$$\int_0^{+\infty} |K(x, 1)| x^{-\frac{1}{p}} dx \lesssim 1 \quad \text{or} \quad \int_0^{+\infty} |K(1, y)| y^{-\frac{1}{q}} dy \lesssim 1.$$

Then

$$\left| \int_0^1 \int_0^1 K(x, y) f(x) g(y) dx dy \right| \lesssim \|f\|_{L^p([0, 1])} \|g\|_{L^q([0, 1])}.$$

In [5], one can find a proof for the case when the integrals that define the bilinear form are taken over $[0, +\infty[$. We treat this slightly trickier case of finite range without using the result in [5].

Proof. Let $Q(f, g) = \int_0^1 \int_0^1 K(x, y) f(x) g(y) dx dy$. Using a change of variables, $x \rightarrow y.u$, and exploiting the homogeneity of the kernel we have

$$Q(f, g) = \int_0^1 y g(y) \int_0^{\frac{1}{y}} K(y.u, y) f(y.u) du dy = \int_0^1 g(y) \int_0^{\frac{1}{y}} K(u, 1) f(y.u) du dy.$$

By Fubini's theorem we may write

$$Q(f, g) = \int_0^1 K(u, 1) \int_0^1 f(y.u) g(y) dy du + \int_1^{+\infty} K(u, 1) \int_0^{\frac{1}{u}} f(y.u) g(y) dy du. \quad (41)$$

But by Hölder's inequality we have

$$\begin{aligned} \left| \int_0^1 f(y.u) g(y) dy \right| &\leq \left(\int_0^1 |f(y.u)|^p dy \right)^{\frac{1}{p}} \left(\int_0^1 |g(y)|^q dy \right)^{\frac{1}{q}} \\ &= u^{-\frac{1}{p}} \left(\int_0^u |f(x)|^p dx \right)^{\frac{1}{p}} \|g\|_{L^q([0,1])} \leq u^{-\frac{1}{p}} \|f\|_{L^q([0,1])} \|g\|_{L^q([0,1])} \end{aligned}$$

for all $0 < u < 1$. Similarly

$$\begin{aligned} \left| \int_0^{\frac{1}{u}} f(y.u) g(y) dy \right| &\leq \left(\int_0^{\frac{1}{u}} |f(y.u)|^p dy \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{u}} |g(y)|^q dy \right)^{\frac{1}{q}} \\ &= u^{-\frac{1}{p}} \left(\int_0^1 |f(x)|^p dx \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{u}} |g(y)|^q dy \right)^{\frac{1}{q}} \leq u^{-\frac{1}{p}} \|f\|_{L^q([0,1])} \|g\|_{L^q([0,1])} \end{aligned}$$

for all $1 < u < +\infty$. Using the last two inequalities together with the triangle inequality in (41) we get

$$\begin{aligned} |Q(f, g)| &\leq \|f\|_{L^q([0,1])} \|g\|_{L^q([0,1])} \left(\int_0^1 |K(u, 1)| u^{-\frac{1}{p}} du + \int_1^{+\infty} |K(u, 1)| u^{-\frac{1}{p}} du \right) \\ &\lesssim \|f\|_{L^q([0,1])} \|g\|_{L^q([0,1])}, \quad \text{when } \int_0^{+\infty} |K(x, 1)| x^{-\frac{1}{p}} dx \lesssim 1. \end{aligned}$$

When $\int_0^{+\infty} |K(1, y)| y^{-\frac{1}{q}} dy \lesssim 1$ the assertion follows analogously. \square

Remark 4. If $K(x, y) = (x + y)^{-1}$ in Lemma 8 we get Hilbert's inequality.

Lemma 9. Let $f, g \in L^2([-1, 1])$. Then

$$\int_{-1}^1 \int_{-1}^1 \frac{|f(x)| |g(y)|}{1 + N |x^2 - y^2|} dx dy \lesssim \frac{1}{\sqrt{N}} \|f\|_{L^2([-1,1])} \|g\|_{L^2([-1,1])}, \quad (42)$$

$$\int_{-1}^1 \int_{-1}^1 \frac{|f(x)| |g(y)|}{\sqrt{|x^2 - y^2|}} dx dy \lesssim \|f\|_{L^2([-1,1])} \|g\|_{L^2([-1,1])}. \quad (43)$$

Proof. Beginning with the estimate (42), Remark 3 suggests estimating

$$\int_0^1 \int_0^1 \frac{|f(\pm x)||g(\pm y)|}{1 + N |x^2 - y^2|} dx dy. \text{ Let } W_N(f, g) := \int_0^1 \int_0^1 \frac{|f(x)||g(y)|}{1 + N |x^2 - y^2|} dx dy.$$

If $x, y \geq 0$ and $|x - y| \gg 1/\sqrt{N}$ then we also have $x + y \gg 1/\sqrt{N}$ and consequently $N|x^2 - y^2| \gg 1$. Therefore

$$\begin{aligned} W_N(f, g) &\approx \int \int_{\substack{0 \leq x, y \leq 1, \\ |x-y| \lesssim 1/\sqrt{N}}} \frac{|f(x)||g(y)|}{1 + N |x^2 - y^2|} dx dy + \int \int_{\substack{0 \leq x, y \leq 1, \\ |x-y| \gg 1/\sqrt{N}}} \frac{|f(x)||g(y)|}{1 + N |x^2 - y^2|} dx dy \\ &\lesssim \int \int_{\substack{0 \leq x, y \leq 1, \\ |x-y| \lesssim 1/\sqrt{N}}} |f(x)||g(y)| dx dy + \frac{1}{N} \int \int_{\substack{0 \leq x, y \leq 1, \\ |x-y| \gg 1/\sqrt{N}}} \frac{|f(x)||g(y)|}{|x^2 - y^2|} dx dy \\ &\lesssim \int_0^1 \int_0^1 \chi_N(|x - y|) |f(x)||g(y)| dx dy + \frac{1}{\sqrt{N}} \int_0^1 \int_0^1 \frac{|f(x)||g(y)|}{x + y} dx dy \end{aligned} \quad (44)$$

where χ_N is the characteristic function of the interval $[0, 1/\sqrt{N}]$. By Young's inequality we have

$$\int_0^1 \int_0^1 \chi_N(|x - y|) |f(x)||g(y)| dx dy \leq \frac{1}{\sqrt{N}} \|f\|_{L^2([0,1])} \|g\|_{L^2([0,1])}. \quad (45)$$

And by Hilbert's inequality

$$\int_0^1 \int_0^1 \frac{|f(x)||g(y)|}{x + y} dx dy \lesssim \|f\|_{L^2([0,1])} \|g\|_{L^2([0,1])}. \quad (46)$$

Using (45) together with (46) in (44) we obtain

$$\int_0^1 \int_0^1 \frac{|f(x)||g(y)|}{1 + N |x^2 - y^2|} dx dy \lesssim \frac{1}{\sqrt{N}} \|f\|_{L^2([0,1])} \|g\|_{L^2([0,1])}.$$

In obtaining (44), we worked only on the kernel of W_N . It is therefore easy to see that replacing the function $x \rightarrow f(x)$ by the function $x \rightarrow f(-x)$ or $y \rightarrow g(y)$ by $y \rightarrow g(-y)$ then repeating the routine above eventually leads to the estimate

$$\int_0^1 \int_0^1 \frac{|f(\pm x)||g(\pm y)|}{1 + N |x^2 - y^2|} dx dy \lesssim \frac{1}{\sqrt{N}} \|f\|_{L^2([-1,1])} \|g\|_{L^2([-1,1])}.$$

This proves (42). Taking advantage of Remark 3 again and arguing like before, it suffices to

estimate $V(f, g) = \int_0^1 \int_0^1 \frac{|f(x)| |g(y)|}{\sqrt{|x^2 - y^2|}} dx dy$. Since $\int_0^{+\infty} \frac{dz}{\sqrt{z} \sqrt{|1 - z^2|}} \approx 1$,

a direct application of Lemma 8 then gives $V(f, g) \lesssim \|f\|_{L^2([0,1])} \|g\|_{L^2([0,1])}$. \square

We are now ready to prove (34). We do this for each of the cases $k = 1$ and $k = 2$ separately.

The phase is linear in s ($k = 1$):

Let ψ be a nonnegative smooth cutoff function such that $\text{supp } \psi \subset]-1, 2[$ and $\psi(s) = 1$ on $[0, 1]$. Since $|T_N^{j,1} f|^2 = T_N^{j,1} f \overline{T_N^{j,1} f}$. Then

$$\begin{aligned} \|T_N^{j,1} f\|_{L^2([0,1])}^2 &= \int_0^1 |T_N^{j,1} f(s)|^2 ds \leq \int_{\mathbb{R}} \psi(s) |T_N^{j,1} f(s)|^2 ds \\ &= \int_{\mathbb{R}} \psi(s) T_N^{j,1} f(s) \overline{T_N^{j,1} f(s)} ds = \int_{\mathbb{R}} \psi(s) \int_B \int_B e^{iN(|x|^j - |y|^j)s} f(x) \overline{f(y)} dx dy ds. \end{aligned}$$

Let $f \in L^2(B)$. Applying Fubini's theorem we get

$$\|T_N^{j,1} f\|_{L^2([0,1])}^2 \leq \int_B \int_B K_N^{j,1}(x, y) f(x) \overline{f(y)} dx dy. \quad (47)$$

In the light of the estimate (35) of Lemma 6, it follows that

$$\|T_N^{j,1} f\|_{L^2([0,1])}^2 \lesssim \int_B \int_B \frac{|f(x)| |f(y)|}{1 + N^2 (|x|^j - |y|^j)^2} dx dy. \quad (48)$$

Since $\int_0^1 \frac{dz}{1 + N^2 z^2} \approx \frac{1}{N}$, then, applying Lemma 7 with $h(z) = (1 + N^2 z^2)^{-1}$ to the

estimate (48), we obtain

$$\|T_N^{j,1} f\|_{L^2([0,1])} \lesssim \frac{1}{\sqrt{N}} \|f\|_{L^2(B)}, \quad \text{for all dimensions } n \geq j. \quad (49)$$

To finish this case, it remains to estimate $T^{2,1}f$ in the dimension $n = 1$. In view of (35) and (47), we have

$$\|T_N^{2,1}f\|_{L^2([0,1])}^2 \lesssim \int_{-1}^1 \int_{-1}^1 \frac{|f(x)||f(y)|}{1+N|x^2-y^2|} dx dy.$$

Hence, by (42) of Lemma 9,

$$\|T_N^{2,1}f\|_{L^2([0,1])} \lesssim \frac{1}{N^{1/4}} \|f\|_{L^2([-1,1])}. \quad (50)$$

The phase is quadratic in s ($k = 2$):

For $f \in L^2(B)$, using Fubini's theorem then employing the estimate (36) implies

$$\|T_N^{j,2}f\|_{L^2([0,1])}^2 = \int_B \int_B K_N^{j,2}(x,y) f(x) \overline{f(y)} dx dy \lesssim G_N^j(f) + H_N^j(f) \quad (51)$$

where

$$G_N^j(f) = \int_B \int_B \frac{|f(x)||f(y)|}{1+\sqrt{N}\sqrt{||x|^j-|y|^j|}} dx dy,$$

$$H_N^j(f) = \int_B \int_B \frac{|f(x)||f(y)|}{1+N||x|^j-|y|^j|} dx dy.$$

Since $\int_0^1 \frac{dz}{1+\sqrt{N}\sqrt{z}} \approx \frac{1}{\sqrt{N}}$, $\int_0^1 \frac{dz}{1+Nz} = o\left(\frac{1}{\sqrt{N}}\right)$, as $N \rightarrow +\infty$,

then applying Lemma 7 to both $G_N^j(f)$ and $H_N^j(f)$ gives the estimate

$$G_N^j(f) + H_N^j(f) \lesssim \frac{1}{\sqrt{N}} \|f\|_{L^2(B)}^2, \quad n \geq j. \quad (52)$$

It remains to control $G_N^2(f)$ and $H_N^2(f)$ in the dimension $n = 1$. But when $n = 1$,

$$\begin{aligned} G_N^2(f) &= \int_{-1}^1 \int_{-1}^1 \frac{|f(x)||f(y)|}{1+\sqrt{N}\sqrt{|x^2-y^2|}} dx dy \\ &\leq \frac{1}{\sqrt{N}} \int_{-1}^1 \int_{-1}^1 \frac{|f(x)||f(y)|}{\sqrt{|x^2-y^2|}} dx dy \lesssim \frac{1}{\sqrt{N}} \|f\|_{L^2([-1,1])}^2 \quad \text{by (43) of Lemma 9.} \end{aligned}$$

An identical estimate holds for $H_N^2(f)$ in the dimension $n = 1$ because of (42). Combining this with (52) and using them in (51) yields

$$\| T_N^{j,2} f \|_{L^2([0,1])} \lesssim \frac{1}{N^{1/4}} \| f \|_{L^2(B)} . \quad (53)$$

Finally, bringing the estimates (49), (50) and (53) together results in (34).

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